Implementing Stochastic Volatility in DSGE Models: A Comment ☆

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Abstract

We highlight a state variable misspecification with one accepted method to implement stochastic volatility (SV) in DSGE models when transforming the nonlinear state-innovation dynamics to its linear representation. Although the technique is more efficient numerically, we show that it is not exact but only serves as an approximation when the magnitude of SV is small. Not accounting for this approximation error may induce substantial spurious volatility in macroeconomic series, which could lead to incorrect inference about the performance of the model. We also show that, by simply lagging and expanding the state vector, one can obtain the correct state-space specification. Finally, we validate our augmented implementation approach against an established alternative through numerical simulation.

Keywords: Dynamic equilibrium economies; Stochastic volatility; Perturbation; Matlab code

JEL Classification: C63; C68; E37

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1 Introduction

Stochastic volatility (or SV) has become an important ingredient for many dynamic equilibrium models.[1] These models in general show that stochastic volatility, manifested in different source of exogenous shocks, can generate realistic comovements in macroeconomic aggregates consistent with economic intuition and data. This is a nontrivial development for general equilibrium modeling in the aftermath of the 2008 crisis, during which heightened economic and policy uncertainty contributed to a prolonged recovery period.

In an early paper that administers SV in the DSGE model, Andreasen (2012) demonstrates how any model with non-linearities between state variables and innovations, such as those typically encountered in stochastic volatility dynamics, may be rewritten into a standard state-space form where innovations only enter linearly. This allows researchers to employ standard perturbation methods to obtain Taylor series approximations (potentially to arbitrarily high order) of policy functions in DSGE models with stochastic volatility.[2]

To operationalize the perturbation approach, one still needs to specify the state variables in the model.[3] For each exogenous variable x featuring stochastic volatility, assuming they are first order Markovian, it is standard to specify four state variables: the value (x_{t-1}), the level of conditional volatility of x ($\sigma_{x,t-1}$), the innovation to the level ($\epsilon_{x,t}$), and the innovation to the conditional volatility ($\epsilon_{\sigma,t}$). Thus, each structural shock featuring stochastic volatility leads to two additional states.[4] Interestingly, the replication files associated with the economy described in Andreasen (2012) show an implementation of the stochastic volatility process that allows one to solve the model with fewer state variables than the four described above. Since state variable reduction is always desirable

[1]For an incomplete list of recent literature, see Bloom (2009), Fernández-Villaverde et al. (2011), Bloom et al. (2012), Fernández-Villaverde et al. (2015) and Basu and Bundick (2017). In a related paper, Bretscher et al. (2017) argue that increased level of risk aversion can further amplify the effectiveness of stochastic volatility shocks.

[2]See Judd (1998). Many publicly available packages implement perturbation methods, see e.g. Dynare (Adjemian, Bastani, Juillard, Karamé, Maih, Mihoubi, Perendia, Pfeifer, Ratto and Villemot, 2011), the library of numerical routines developed by Schmitt-Grohe and Uribe (2004) and recently extended by Andreasen et al. (2017), and the AIM software routines developed by Swanson et al. (2006).

[3]Note that some packages, like Dynare, automatically specify the state variables.

[4]This is the way Dynare implements SV. See also Fernández-Villaverde et al. (2015).

in numerical methods, the Andreasen (2012) approach is appealing.

This note shows that this reduction in state variables is generally not exact, but it holds only in specific situations. In particular, we show that the set of state variables needs to be conveniently lagged and expanded in order to obtain a valid state-space representation to which one can apply perturbation methods. Although our correction is inconsequential for the results in Andreasen (2012) due to the negligible amount of stochastic volatility employed in his calibration, not using the proper formulation of the state-space may generate close-to-explosive path in other settings for future work. This note thus raises a warning flag to using the "short-cut" described in Andreasen (2012) when implementing SV in DSGE models.

The rest of the paper proceeds as follows. Section 2 describes four alternative ways to implement SV in DSGE models. Section 3 investigates the quantitative implications of the various implementations in two alternative economies featuring stochastic volatility. Throughout the paper we make use of the third and higher order perturbation capacities offered by the set of MATLAB programs developed by Schmitt-Grohe and Uribe (2004), Andreasen (2012), and Levintal (2017).

2 Dynamic Equilibrium Models with Stochastic Volatility

2.1 The General Model

Following Schmitt-Grohe and Uribe (2004), we study models with equilibrium conditions of the form:

$$E_t \left[\mathbf{f} \left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t \right) \right] = \mathbf{0} \ ,$$

where the state vector $\mathbf{x}_t$ has dimension $n_x \times 1$, and the vector $\mathbf{y}_t$ with dimension $n_y \times 1$ contains all the control variables. The vector $\mathbf{x}_t$ is partitioned as $\left[\mathbf{x}'_{1,t}, \mathbf{x}'_{2,t} \right]'$ where $\mathbf{x}_{1,t}$ contains endogenous state variables and $\mathbf{x}_{2,t}$ denotes exogenous state variables. The dimensions of these vectors are $n_{x_1} \times 1$ and $n_{x_2} \times 1$, respectively, where $n_{x_1} + n_{x_2} = n_x$. It is further assumed that

$$\mathbf{x}_{2,t+1} = \mathbf{\Gamma}\left(\mathbf{x}_{2,t}\right) + \Lambda\tilde{\boldsymbol{\eta}}\boldsymbol{\varepsilon}_{t+1} \ , \tag{1}$$

where $\Lambda \geq 0$ is an auxiliary perturbation parameter for the structural innovations $\boldsymbol{\varepsilon}_{t+1}$ with dimension $n_\varepsilon \times 1$.

The solution to these class of models – if one exists – is characterized by a policy function describing the evolution of the endogenous state variables and policy functions describing the law of motion of the observables, namely

$$\begin{aligned}\mathbf{y}_t &= \mathbf{g}(\mathbf{x}_t; \Lambda) \ , \\ \mathbf{x}_{t+1} &= \mathbf{h}(\mathbf{x}_t; \Lambda) + \Lambda\boldsymbol{\eta}\boldsymbol{\varepsilon}_{t+1}\end{aligned} \tag{*}$$

with

$$\boldsymbol{\eta} \equiv \begin{bmatrix} \mathbf{0}_{n_{x_1} \times n_\varepsilon} \\ \tilde{\boldsymbol{\eta}} \end{bmatrix} .$$

As highlighted in Andreasen (2012) a system with non-linearities between state variables and innovations can be rewritten into an extended system with only linear innovations, i.e. the assumption that innovations only enter linearly in (1) is without loss of generality.

Next we illustrate how to rewrite the non-linear dynamics implied by stochastic volatility in a form like (1) containing only linear innovations. The derivation follows closely the steps described in Andreasen (2012)–Appendix A. For each step in the derivation we also provide reference to the specific lines in the associated computer code of the Andreasen (2012) replication files, see Appendix A.

2.2 The Stochastic Volatility Process

As an illustration, consider a neoclassical growth model with stochastic volatility. Using standard notation, the equilibrium conditions are given by:

$$\begin{aligned}C_t^{-\gamma} &= \beta E_t\left[C_{t+1}^{-\gamma}\left(A_{t+1}\alpha K_{t+1}^{\alpha-1} + 1 - \delta\right)\right] \\ C_t + K_{t+1} &= A_t K_t^\alpha + (1-\delta)\, K_t \ ,\end{aligned}$$

so that in the notation of the previous section we have $y_t = C_t$ and $x_{1,t} = K_t$. Notice K_t is determined in period $t-1$ following the convention of Schmitt-Grohe and Uribe (2004). Let's assume that the exogenous level of productivity $x_{2,t} = A_t$ evolves according to[5]

$$\log(A_{t+1}) = \rho_a \log(A_t) + \sigma_{a,t+1}\varepsilon_{a,t+1} \tag{2}$$

$$\log(\sigma_{a,t+1}) = \rho_\sigma \log(\sigma_{a,t}) + \varepsilon_{\sigma,t+1} \ , \tag{3}$$

where $\varepsilon_{a,t} \sim N(0,1)$ and $\varepsilon_{\sigma,t} \sim N(0, \mathrm{Var}\,(\varepsilon_{\sigma,t}))$.
Andreasen (2012)–Appendix A shows that these equations describing the SV process can be equivalently rewritten as:[6]

$$\log(A_t) = \sigma_{a,t} \log V_t \tag{4}$$

$$\log V_{t+1} = \rho_a \frac{\sigma_{a,t}}{\sigma_{a,t+1}} \times \log V_t + \varepsilon_{a,t+1} \tag{5}$$

$$\log(\sigma_{a,t+1}) = \rho_\sigma \log(\sigma_{a,t}) + \varepsilon_{\sigma,t+1} \ . \tag{6}$$

As is standard in the literature, we use log-transformed variables, namely we replace

$$A_t, V_t, \ \sigma_{a,t} \longrightarrow e^{a_t}, e^{v_t}, \ e^{\tilde{\sigma}_{a,t}} \ ,$$

where lowercase letters denote log variables, i.e. $a_t = \log A_t$, $v_t = \log V_t$ and $\tilde{\sigma}_{a,t} = \log \sigma_{a,t}$.[7] Eqs. (4)–(6) now read:

$$a_t = e^{\tilde{\sigma}_{a,t}} v_t \tag{7}$$

$$v_{t+1} = \rho_a \frac{e^{\tilde{\sigma}_{a,t}}}{e^{\tilde{\sigma}_{a,t+1}}} \times v_t + \varepsilon_{a,t+1} \tag{8}$$

$$\tilde{\sigma}_{a,t+1} = \rho_\sigma \tilde{\sigma}_{a,t} + \varepsilon_{\sigma,t+1} \tag{9}$$

[5] We neglect the steady state to ease notation.
[6] See lines 61 – 65 and 73 – 74 in `NK_Rotemberg_SV_model.m`.
[7] The log-transformation is also implemented in the MATLAB code. See line 92 in `NK_Rotemberg_SV_model.m`.

2.3 Definition of State Variables: the "Direct" Specification

So far we acknowledge (one of) the contribution in Andreasen (2012) that shows how the system in Eqs. (2)–(3) which is characterized by non-linearities between state variables and innovations can be rewritten as in Eqs. (7)–(9) where innovations only enter linearly.

However, to eventually obtain a solution via perturbation method we need to list the state variables. Inspection of the MATLAB code provided by Andreasen (2012) suggests to use[8]

$$\mathbf{x}_{2,t} = [v_t, \ \tilde{\sigma}_{a,t}]' \ . \tag{10}$$

We dub this choice of state variables the "Direct" specification.

Next, we show that this particular choice yields a system of equations which is *not* in the form of (*). To see this define also $y_t = a_t$. It is straightforward to see that, given the choice of state variables in Eq. (10), $y_t = g(x_t;\Lambda)$ is satisfied with $g(a,b;\Lambda) = e^a \times b$, and $a \equiv \sigma_{a,t}$, $b \equiv v_t$. The state equation in (*) becomes

$$\mathbf{x}_{2,t+1} = \begin{bmatrix} v_{t+1} \\ \tilde{\sigma}_{a,t+1} \end{bmatrix} = \begin{bmatrix} \underbrace{\rho_a \frac{e^{\tilde{\sigma}_{a,t}}}{e^{\tilde{\sigma}_{a,t+1}}} \times v_t}_{h_1(a,\ b)=\rho_a e^{(a-a')}\times b} \\ \underbrace{\rho_\sigma \tilde{\sigma}_{a,t}}_{h_2(a,b)=\rho_\sigma \times a + 0 \times b} \end{bmatrix} + \Lambda \begin{bmatrix} \varepsilon_{a,t+1} \\ \varepsilon_{\sigma,t+1} \end{bmatrix} \tag{11}$$

where $a' \equiv \tilde{\sigma}_{a,t+1}$.

Clearly not all the variables on the right hand side of (11) are adapted at time t. So we exploit the dynamics of $\mathbf{x}_{2,t+1}$ to replace these $(t+1)$–variables with their time t value. We have

$$\begin{aligned} \rho_a \frac{e^{\tilde{\sigma}_{a,t}}}{e^{\tilde{\sigma}_{a,t+1}}} \times v_t &= \rho_a e^{\tilde{\sigma}_{a,t}-\tilde{\sigma}_{a,t+1}} \times v_t \\ &= \rho_a e^{\tilde{\sigma}_{a,t}(1-\rho_\sigma)-\varepsilon_{\sigma,t+1}} \times v_t \ , \end{aligned}$$

where $\tilde{\sigma}_{a,t} - \tilde{\sigma}_{a,t+1} = \tilde{\sigma}_{a,t}(1-\rho_\sigma) - \varepsilon_{\sigma,t+1}$. Now observe that $\varepsilon_{\sigma,t+1}$ is not a state variable in the "Direct" specification, see Eq. (10). It cannot even be singled out because of the interaction term, i.e. $e^{-\varepsilon_{\sigma,t+1}} \times v_t$. We conclude that the "Direct" specification leads to a

[8] See lines 88 – 89 in `NK_Rotemberg_SV_model.m`.

system of equations which is *not* in the form described in Eq. (*).

2.4 Definition of State Variables: Fixing the "Direct" Specification

We show that to put Eqs. (8)-(9) in the form of Eq. (*) we must expand the state vector. First lag Eqs. (8)-(9),

$$v_t = \rho_a \frac{e^{\tilde{\sigma}_{a,t-1}}}{e^{\tilde{\sigma}_{a,t}}} \times v_{t-1} + \varepsilon_{a,t} \tag{12}$$

$$\tilde{\sigma}_{a,t} = \rho_\sigma \tilde{\sigma}_{a,t-1} + \varepsilon_{\sigma,t} \ . \tag{13}$$

Next, specify the following state vector

$$\mathbf{x}_{2,t} = \left[v_{t-1},\ \tilde{\sigma}_{a,t-1},\ \varepsilon_{a,t},\ \varepsilon_{\sigma,t} \right]' \ . \tag{14}$$

We dub this choice of state variables the "Lagged Direct" specification.
Importantly, Eqs. (12)–(13) are in the form described in Eq. (*):

$$\mathbf{x}_{2,t+1} = \begin{bmatrix} v_t \\ \tilde{\sigma}_{a,t} \\ \varepsilon_{a,t+1} \\ \varepsilon_{\sigma,t+1} \end{bmatrix} = \begin{bmatrix} \rho_a \frac{e^{\tilde{\sigma}_{a,t-1}}}{e^{\tilde{\sigma}_{a,t}}} \times v_{t-1} + \varepsilon_{a,t} \\ \rho_\sigma \tilde{\sigma}_{a,t-1} + \varepsilon_{\sigma,t} \\ 0 \\ 0 \end{bmatrix} + \Lambda \begin{bmatrix} 0 \\ 0 \\ u_{a,t+1} \\ u_{\sigma,t+1} \end{bmatrix}$$

2.5 The Fernández-Villaverde et al. (2015) Specification

Fernández-Villaverde et al. (2015) propose an implementation of stochastic volatility in DSGE models which is alternative to Andreasen (2012) (see Section 2.2 and 2.3). Recall Eqs. (2)–(3) reported here for reader convenience

$$\log(A_{t+1}) = \rho_a \log(A_t) + \sigma_{a,t+1}\varepsilon_{a,t+1}$$

$$\log(\sigma_{a,t+1}) = \rho_\sigma \log(\sigma_{a,t}) + \varepsilon_{\sigma,t+1} \ .$$

Fernández-Villaverde et al. (2015) specification consists of the following steps. Lag the above system of equations

$$\log(A_t) = \rho_a \log(A_{t-1}) + \sigma_{a,t}\varepsilon_{a,t} \tag{15}$$

$$\log(\sigma_{a,t}) = \rho_\sigma \log(\sigma_{a,t-1}) + \varepsilon_{\sigma,t} \tag{16}$$

and log-transform the variables,

$$A_t,\ \sigma_{a,t} \longrightarrow e^{a_t},\ e^{\tilde{\sigma}_{a,t}} \ ,$$

so that Eqs. (15)–(16) now read

$$a_t = \rho_a a_{t-1} + e^{\tilde{\sigma}_{a,t}}\varepsilon_{a,t} \tag{17}$$

$$\tilde{\sigma}_{a,t} = \rho_\sigma \tilde{\sigma}_{a,t-1} + \varepsilon_{\sigma,t} \ . \tag{18}$$

Fernández-Villaverde et al. (2015) choose the following state vector:

$$\mathbf{x}_{2,t} = [a_{t-1},\ \tilde{\sigma}_{a,t-1},\ \varepsilon_{a,t},\ \varepsilon_{\sigma,t}]' \ . \tag{19}$$

We can finally verify that Eqs. (17)–(18) are in the form described in Eq. (*):

$$\mathbf{x}_{2,t+1} = \begin{bmatrix} a_t \\ \tilde{\sigma}_{a,t} \\ \varepsilon_{a,t+1} \\ \varepsilon_{\sigma,t+1} \end{bmatrix} = \begin{bmatrix} \underbrace{\rho_a a_{t-1} + e^{\tilde{\sigma}_{a,t}}\varepsilon_{a,t}}_{h_1(a,\ b,\ c,\ d)=\rho_a a + e^{b'} c} \\ \underbrace{\rho_\sigma \tilde{\sigma}_{a,t-1} + \varepsilon_{\sigma,t}}_{h_2(a,\ b,\ c,\ d)=0\times a+\rho_\sigma\times b+0\times c+d} \\ 0 \\ 0 \end{bmatrix} + \Lambda \begin{bmatrix} 0 \\ 0 \\ u_{a,t+1} \\ u_{\sigma,t+1} \end{bmatrix}$$

We dub this choice of state variables the "FGR" specification.

2.6 The Fernández-Villaverde and Levintal (2017) Specification

Fernández-Villaverde and Levintal (2017) propose yet another implementation of stochastic volatility which is similar to the implementation of Fernández-Villaverde et al. (2015)

discussed above. In contrast to Fernández-Villaverde et al. (2015), however, the state vector $\mathbf{x}_{2,t}$ only consists of just three variables. In particular, the state vector is defined as follows:

$$\mathbf{x}_{2,t} = \left[a_{t-1},\ \tilde{\sigma}_{a,t},\ \varepsilon_{a,t}\right]' . \tag{20}$$

With this choice, the state equation can still be written in the form described in Eq. (*):

$$\mathbf{x}_{2,t+1} = \begin{bmatrix} a_t \\ \tilde{\sigma}_{a,t+1} \\ \varepsilon_{a,t+1} \end{bmatrix} = \begin{bmatrix} \underbrace{\rho_a a_{t-1} + e^{\tilde{\sigma}_{a,t}} \varepsilon_{a,t}}_{h_1(a,\ b,\ c) = \rho_a a + e^{b'} c} \\ \underbrace{\rho_\sigma \tilde{\sigma}_{a,t}}_{h_2(a,\ b,\ c) = 0 \times a + \rho_\sigma \times b + 0 \times c} \\ 0 \end{bmatrix} + \Lambda \begin{bmatrix} 0 \\ u_{\sigma,t+1} \\ u_{a,t+1} \end{bmatrix}$$

For obvious reasons, we dub this choice of state variables the "FL" specification.
Due to the reduction in the size of the state vector, the "FL" implementation is computationally more efficient than the "FGR" approach. Importantly, however, the "FL" approach can only be implemented in perturbation packages that allow the exogenous state variable ($\tilde{\sigma}_{a,t}$) to be an autoregressive process, as in Schmitt-Grohe and Uribe (2004) and the extended packages by Andreasen (2012) and Levintal (2017).[9]

2.7 A Brief Comparison of Stochastic Volatility Implementations

We have described four ways to implement SV in DSGE models: the "Direct", the "Lagged Direct", the "FGR", and the "FL" approach (see Section 2.3, 2.4, 2.5, and 2.6, respectively).

The "Direct" and "Lagged Direct" specifications reshuffle the SV equations (2)–(3) in the form suggested by Andreasen (2012), i.e. (7)–(9). The "FGR" and "FL" approaches work directly with Eqs. (2)–(3) instead.

The "Lagged Direct" and "FGR" specifications have the same number of state variables, see (14) and (19). This is higher than the number of state variables in the "Direct"

[9] Since Dynare assumes that the exogenous state variables are i.i.d. shocks, the "FL" approach cannot be implemented.

specification, see Eq. (10). Specifically, if one has n processes with SV, then the number of state variables is higher by $2n$. E.g. in the model economy of Fernández-Villaverde et al. (2011) we have $n = 2$ variables with SV, and 4 more states in the "Lagged Direct" and "FGR" specifications than in the "Direct" one. This consideration makes the "Direct" specification appealing. However, we show in Section (2.3) that the Direct approach is not consistent with the form described in Eq. (*). Finally, in terms of computational efficiency, the "FL" approach is superior to the "Lagged Direct" and the "FGR" specifications. However, as discussed above, the implementation of "FL" requires a perturbation package that allows the exogenous state variables to be defined as autoregressive processes.

The next section quantifies the implications for the various SV specifications in the context of two alternative economies featuring stochastic volatility.

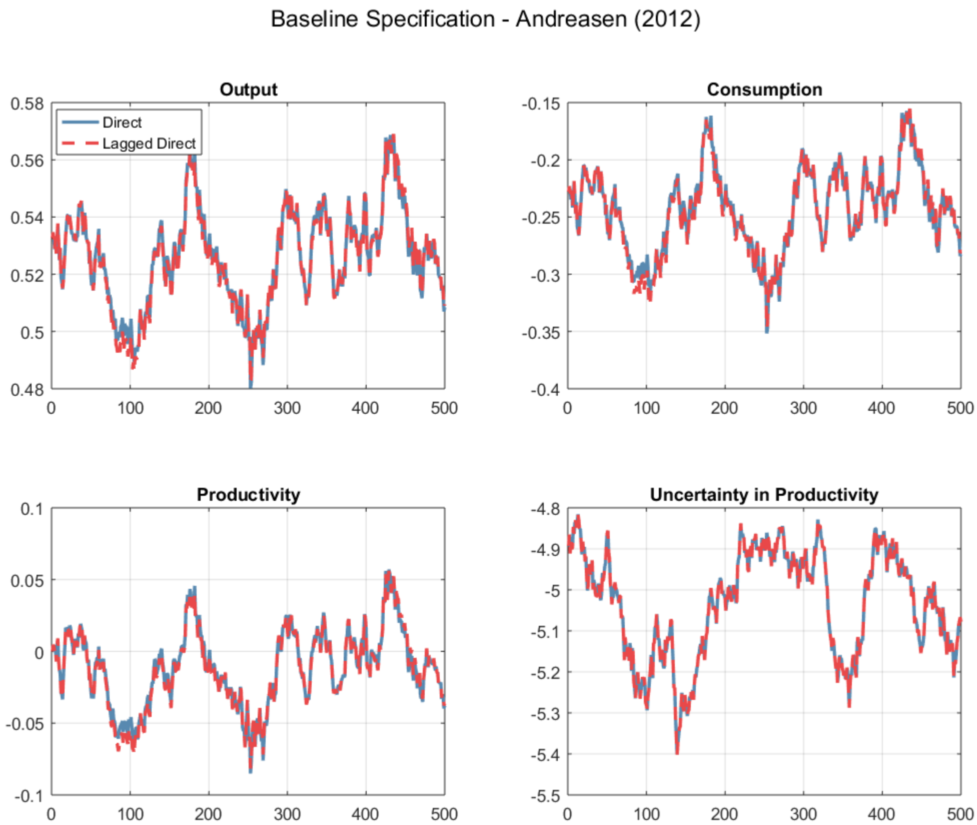
3 A Quantitative Evaluation of the Different SV Specifications

3.1 Volatility in Productivity

We first consider the model economy described in Andreasen (2012). Our purpose is to verify to what extent the "Direct" specification used in that paper alters its conclusion. To this end, we compare the "Direct" specification against the "Lagged Direct" one. In the interest of space, we do not report results for the "FGR" nor for the more efficient "FL" specification since they both numerically coincide with those obtained from the "Lagged Direct" approach.

Figure 1 compares the paths of output, consumption, productivity level and productivity uncertainty as implied by the "Direct" specification (solid line) with that implied by the "Lagged Direct" specification (dashed line). We observe no significant differences. We also verify that the variance decomposition in the two SV specifications does not change.

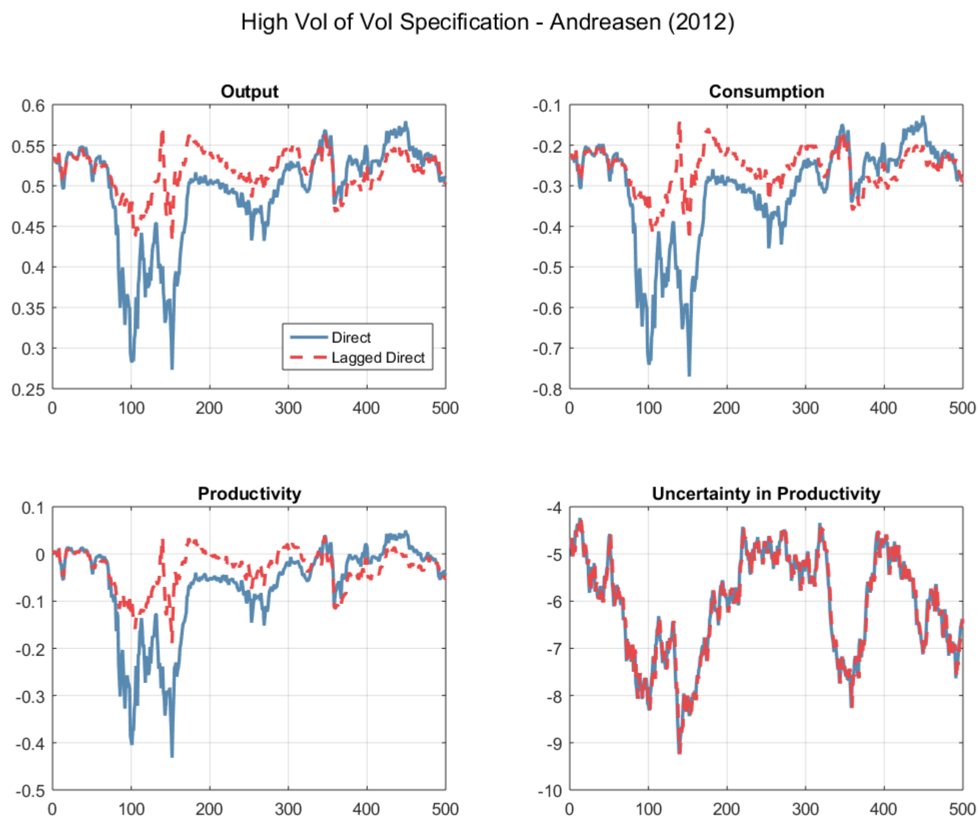
FIGURE 1: ANDREASEN (2012) ECONOMY: SIMULATION.



We simulate the Andreasen (2012) economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values and we follow the results for the deviations of output and consumption (top panels) with respect to the steady state when we have a third-order approximation. The bottom panels display the path for the productivity level (left) and its time-varying volatility (right).

We argue that the two specifications yield similar results due to the chosen value for the vol-of-vol parameter, i.e. the volatility of $\varepsilon_{\sigma,t+1}$. Indeed, Figure 2 shows that the difference between the two alternative specifications gets larger for output, consumption and productivity level as we increase the vol-of-vol parameter from 0.0265 to 0.2265.

FIGURE 2: ANDREASEN (2012) ECONOMY: SIMULATION.



We simulate the Andreasen (2012) economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values except for the vol-of-vol which is set to 0.2265 compared to the original value 0.0265, and we follow the results for the deviations of output and consumption (top panels) with respect to the steady state when we have a third-order approximation. The bottom panels display the path for the productivity level (left) and its time-varying volatility (right).

However, for the range of values commonly used in the literature,[10] the "Lagged Direct" and "Direct" specifications yield extremely similar results for the model economy described in Andreasen (2012). Hence we conclude that using the correct "Lagged Direct" specification does not alter the conclusion in Andreasen (2012).

[10]The literature has found values of the vol-of-vol in productivity ranging from 0.01 to 0.03, see e.g. Justiniano and Primiceri (2008) and Kung (2015).

3.2 Volatility in Interest Rates

We next implement the "Direct" and "Lagged Direct" SV specifications in the model economy of Fernández-Villaverde et al. (2011). In this model, a country faces a real interest rate, r_t, on loans denominated in US dollars. This real rate is decomposed into the international risk-free real rate plus a country–specific spread:

$$r_t = r + \varepsilon_{tb,t} + \varepsilon_{r,t} \ ,$$

where r is the mean of the international risk-free real rate plus the mean of the country spread. The term $\varepsilon_{tb,t}$, equals the international risk-free real rate subtracted from its mean, and $\varepsilon_{r,t}$ equals the country spread subtracted from its mean. Both $\varepsilon_{tb,t}$ and $\varepsilon_{r,t}$ follow AR(1) processes described by

$$\varepsilon_{tb,t} = \rho_{tb}\varepsilon_{tb,t-1} + e^{\tilde{\sigma}_{tb,t}}u_{tb,t}$$
$$\varepsilon_{r,t} = \rho_r\varepsilon_{r,t-1} + e^{\tilde{\sigma}_{r,t}}u_{r,t} \ ,$$

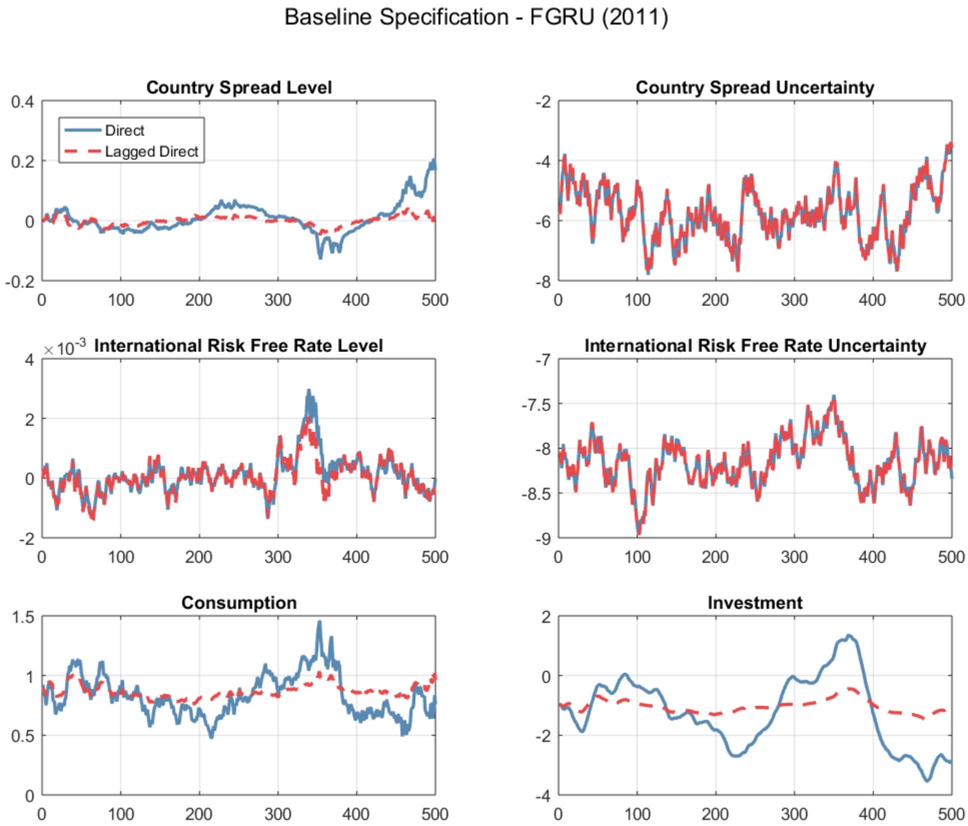
where $u_{r,t}$ and $u_{tb,t}$ are normally distributed random variables with mean zero and unit variance. The process for interest rates displays stochastic volatility. In particular, the standard deviations $\sigma_{tb,t}$ and $\sigma_{r,t}$ follow an AR(1) process:

$$\tilde{\sigma}_{tb,t} = (1 - \rho_{\sigma_{tb}})\,\sigma_{tb} + \rho_{\sigma_{tb}}\tilde{\sigma}_{tb,t-1} + \eta_{tb}u_{\sigma_{tb},t} \qquad (21)$$
$$\tilde{\sigma}_{r,t} = (1 - \rho_{\sigma_r})\,\sigma_r + \rho_{\sigma_r}\tilde{\sigma}_{r,t-1} + \eta_r u_{\sigma_r,t} \ , \qquad (22)$$

where both $u_{\sigma_{tb},t}$ and $u_{\sigma_r,t}$ are normally distributed random variables with mean zero and unit variance.

Figure 3 shows, for various variables of interest, two simulated paths, one obtained under the "Direct" specification and one under the "Lagged Direct" specification. The variables of interest are: the level of the risk-free and country spread interest rates, $\varepsilon_{tb,t}$ and $\varepsilon_{r,t}$ (left panels, top and mid), their volatility $\tilde{\sigma}_{tb,t}$ and $\tilde{\sigma}_{r,t}$ (right panels, top and mid), and the consumption and investment series (bottom panels, left and right). As in the previous subsection, we do not report results for the "FGR" nor the "FL" specification since they numerically coincides with those obtained from the "Lagged Direct" approach.

Figure 3: Fernandez-Villaverde et al. (2011) economy: Simulation.

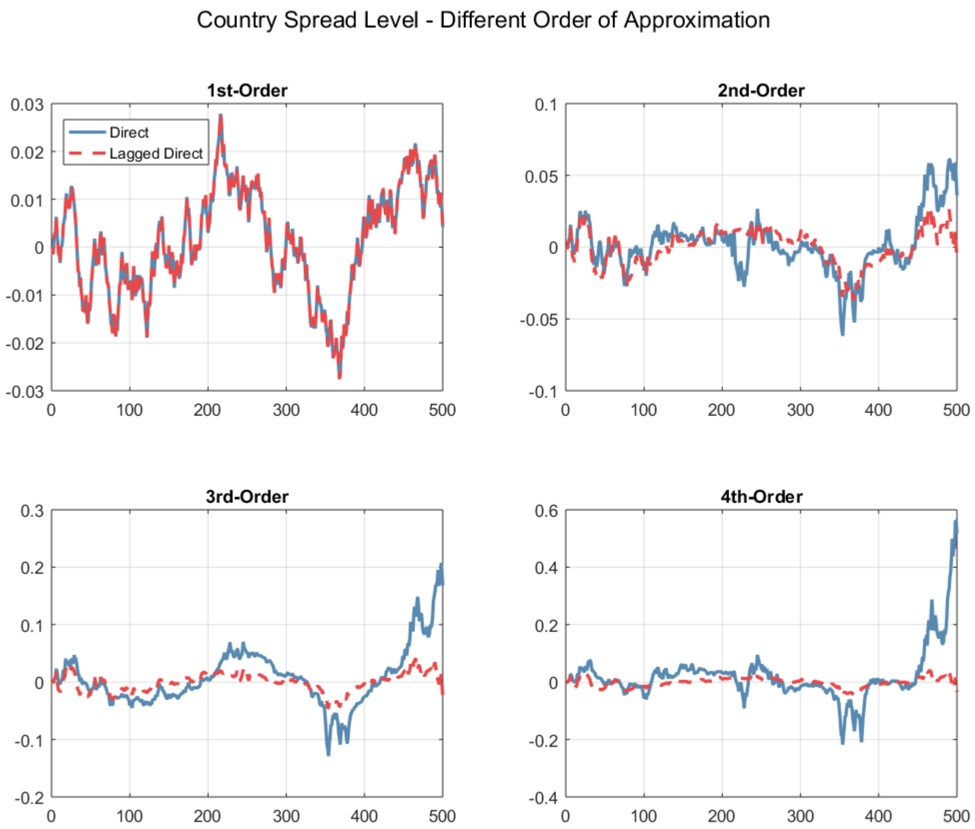


We simulate the Fernandez-Villaverde et al. (2011b) Argentinian economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values and we follow the results for the deviations of consumption and investment (bottom panels) with respect to the steady state when we have a third-order approximation. The top fourth panels display the path for the country spread (top row) and risk-free interest rate (middle row) level (left panels) and volatility (right panels) variables.

Two things are noteworthy. First, the two implementations imply no difference in the path of the time-varying *volatilities*. This is to be expected since each of the (log) volatilities follows a standard AR(1) process which does not involve non-linearities, and thus its implementation is standard. Second, the difference between the path under the "Direct" and "Lagged Direct" specifications is large for the *level* variables, and remarkably so for the consumption and investment series (bottom panels).

Figure 4 shows that the difference induced by the "Direct" and "Lagged Direct" specifications increases with the order of approximation.[11] This is consistent with the literature on perturbation methods showing the necessity of higher order terms for the purpose of capturing effects of stochastic volatility.

Figure 4: Fernandez-Villaverde et al. (2011) economy: Simulation for Different Order of Approximations.

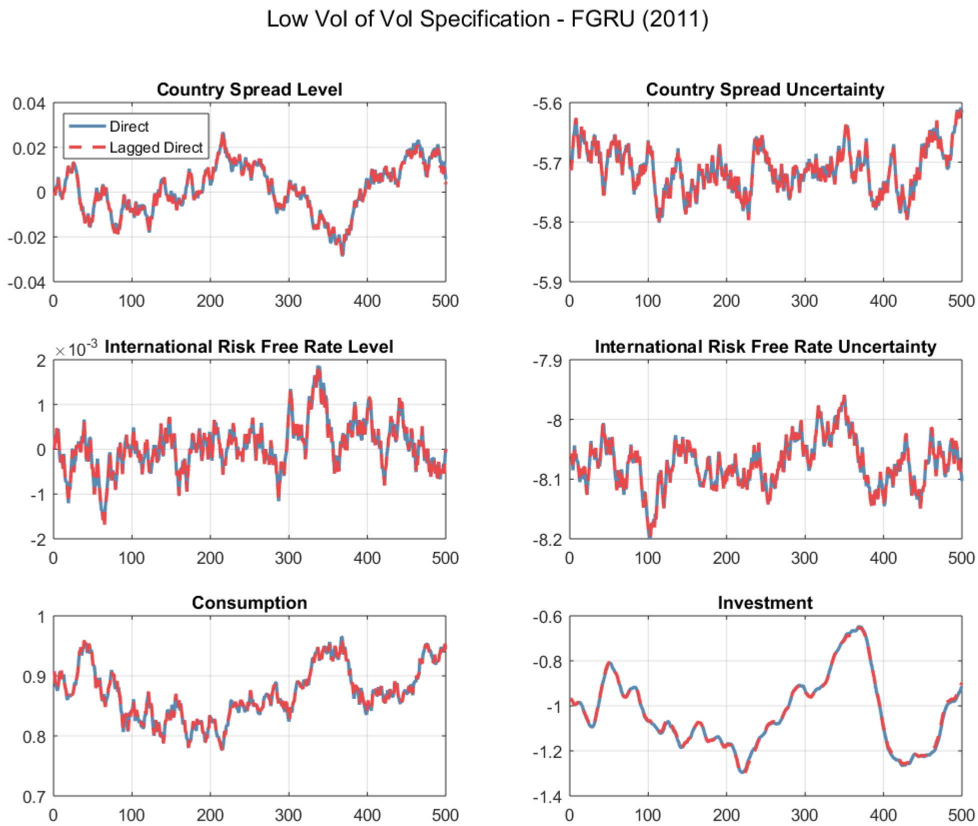


We simulate the Fernandez-Villaverde et al. (2011b) Argentinian economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values and we follow the results for the deviations of consumption and investment (bottom panels) with respect to the steady state when we have a first-, second-, third-, and fourth-order approximation. The top fourth panels display the path for the country spread (top row) and risk-free interest rate (middle row) level (left panels) and volatility (right panels) variables.

[11] den Haan and de Wind (2010) investigate the behavior of higher-order perturbation solutions with and without pruning. To simulate the economy we adopt the pruning scheme proposed by Andreasen et al. (2017). Our conclusion does not change when we instead use an unpruned state-space system.

Finally, Figure 5 shows that these differences disappear as the vol-of-vol parameters, η_r and η_{tb}, are reduced in magnitude. In particular, for this experiment we set both the vol-of-vol parameters of the country spread and the international risk-free rate to 0.02, a value much lower than those used in the original manuscript (namely $\eta_r = 0.46$ and $\eta_{tb} = 0.13$). We conclude the for the small open emerging economies described in Fernández-Villaverde et al. (2011) the "Lagged Direct" specification must be used to achieve the correct conclusions.

FIGURE 5: FERNANDEZ-VILLAVERDE ET AL. (2011) ECONOMY: SIMULATION WITH LOW VOL-OF-VOL.



We simulate the Fernandez-Villaverde et al. (2011b) Argentinian economy for 500 periods (after a period of burn-in to eliminate the effect of initial conditions) at the benchmark calibration parameter values except for the vol-of-vol parameters of the country spread and the international risk-free rate which are both set to 0.02 compared to the original values 0.46 and 0.13, respectively. We then follow the results for the deviations of consumption and investment (bottom panels) with respect to the steady state when we have a third-order approximation. The top fourth panels display the path for the country spread (top row) and risk-free interest rate (middle row) level (left panels) and volatility (right panels) variables.

4 Conclusion

This note discusses subtle issues arising from the implementation of stochastic volatility in the DSGE model described by Andreasen (2012). We show that, to correct the procedure, it is enough to augment the vector of exogenous state variables. Although our corrected version of the SV implementation does not alter the conclusion in Andreasen (2012), we show that for alternative parametrization and economies, the incorrect specification may generate substantial spurious variability in macroeconomic series. The error induced by the incorrect specification is increasing with the order of approximation and with the value of the volatility-of-volatility parameter.

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A The Published Replication Files

This Appendix shows the MATLAB code of the Andreasen (2012) replication file `NK_Rotemberg_SV_model.m` posted at https://sites.google.com/site/mandreasendk/home-1. Section 2.3 highlights the potential issues associated with this implementation.

```
function [f,x,xp,y,yp] = NK_Rotemberg_SV_model
% This function reports the equations for the New Keynesian model with
% Rothenberg prices and SV in f, and it reports the state vector
%(x and xp) and the control vector (y and yp)

  %Define parameters
  syms GAMA NU BETTA ALFA THETA ETA XI DELTA PHI_pai PHI_y RHOA RHOG RHOR RHOSIGA RHOSIGG...
       AA Ass Gss Kss Yss PAIss Rss SIGAss MUZss

%Define variables
syms p_cu  p2_cu  r_cu  y_cu  c_cu  n_cu  evf_cu  pai_cu  a_cu  varsdf_cu ...
     p_cup p2_cup r_cup y_cup c_cup n_cup evf_cup pai_cup a_cup varsdf_cup ...
     r_ba1  va_cu  siga_cu  g_cu  epsR_cu  ...
     r_ba1p va_cup siga_cup g_cup epsR_cup

% Defining muz - ONLY a deterministic trend. For a stochastic trend the
% coding must be changed
muz_cu  = MUZss;
muz_cup = MUZss;

% EQ2: FOC for household with respect to labour
w_cu  = (1-NU)*c_cu/(NU*(1-n_cu));

% EQ4: FOC for frims with respect to labour
mc_cu  = w_cu/((1-THETA)*a_cu*Kss^THETA*n_cu^(-THETA));

% EQ1 The expression for minus the value function
% Here we use the fact that the trend is deterministic
mvf_cup = -(c_cup^(NU*(1-GAMA))*(1-n_cu)^((1-NU)*(1-GAMA))/(1-GAMA)-BETTA*muz_cup^(NU*(1-GAMA))*AA*evf_cup^(1/(1-ALFA)));

% The ratio of lambda_cup/lambda_cu
mu_la_cup = (AA*evf_cu^(1/(1-ALFA))/mvf_cup)^ALFA*muz_cup^(NU*(1-GAMA)-1)...
            *(c_cup/c_cu)^(NU*(1-GAMA)-1)*((1-n_cup)/(1-n_cu))^((1-NU)*(1-GAMA));
```

```
% The equations in f
% The expected value of the value function
f1 = -evf_cu + (mvf_cup/AA)^(1-ALFA);

% EQ3: The one period bond price
f2 = - p_cu + BETTA*mu_la_cup*1/pai_cup;
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% The one period interest rate
f3 = -r_cu + 1/p_cu;

% EQ5: The FOC firms with respect to prices
f4 = - mc_cu + (ETA-1)/ETA ...
         - BETTA*(AA*evf_cu^(1/(1-ALFA))/mvf_cup)^ALFA*muz_cup^(NU*(1-GAMA)) ...
          *(c_cup/c_cu)^(NU*(1-GAMA)-1)*((1-n_cup)/(1-n_cu))^((1-NU)*(1-GAMA)) ...
          *XI/ETA*(pai_cup/PAIss-1)*pai_cup*y_cup/(PAIss*y_cu) ...
         + XI/ETA*(pai_cu/PAIss-1)*pai_cu/PAIss;

% EQ6: The Taylor rule
f5 = -log(r_cu/Rss) +RHOR*log(r_ba1/Rss) + PHI_pai*log(pai_cu/PAIss) + PHI_y*log(y_cu/Yss) + epsR_cu;

% EQ7: The output level
f6 = -y_cu + a_cu*Kss^THETA*n_cu^(1-THETA);

% EQ8: The budget resource constraint
f7 = -y_cu + c_cu + g_cu + DELTA*Kss;

% Defining technolog
f8 = -log(a_cu/Ass) + siga_cu *log(va_cu);

% Technology shocks: the process for va_cu
f9 = -log(va_cup) + RHOA*siga_cu/siga_cup*log(va_cu);

% Defining government spending
f10 = -log(g_cup/Gss) + RHOG*log(g_cu/Gss);

% Monetary policy shocks
f11 = -epsR_cup;

% The process for volatility in technology
f12 = -log(siga_cup/SIGAss) + RHOSIGA*log(siga_cu/SIGAss);

```

```
% The link for r_ba1
f13 = -r_ba1p + r_cu;

% The variance of the nominal stochastic discount factor
f14 = -varsdf_cu + (BETTA*mu_la_cup*1/pai_cup)^2-(1/r_cu)^2;

%Creating function f
f = [f1;f2;f3;f4;f5;f6;f7;f8;f9;f10;f11;f12;f13;f14];

% Define the variables which needs to be log-transformed
y  = [p_cu, r_cu ,y_cu, c_cu, n_cu, evf_cu, pai_cu ,a_cu ];
yp = [p_cup,r_cup,y_cup,c_cup,n_cup,evf_cup,pai_cup,a_cup];
x  = [r_ba1 ,va_cu, siga_cu ,g_cu ];
xp = [r_ba1p,va_cup,siga_cup,g_cup];

%Make f a function of the logarithm of the state and control vector
f = subs(f, [x,y,xp,yp], exp([x,y,xp,yp]));

% Define the vector of controls, y, and states, x
y  = [p_cu, r_cu ,y_cu, c_cu, n_cu, evf_cu, pai_cu ,a_cu ,varsdf_cu];
yp = [p_cup,r_cup,y_cup,c_cup,n_cup,evf_cup,pai_cup,a_cup,varsdf_cup];
x  = [r_ba1 ,va_cu, siga_cu ,g_cu ,epsR_cu];
xp = [r_ba1p,va_cup,siga_cup,g_cup,epsR_cup];
```